Normal approximation

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In Bernoulli distribution, its PDF is

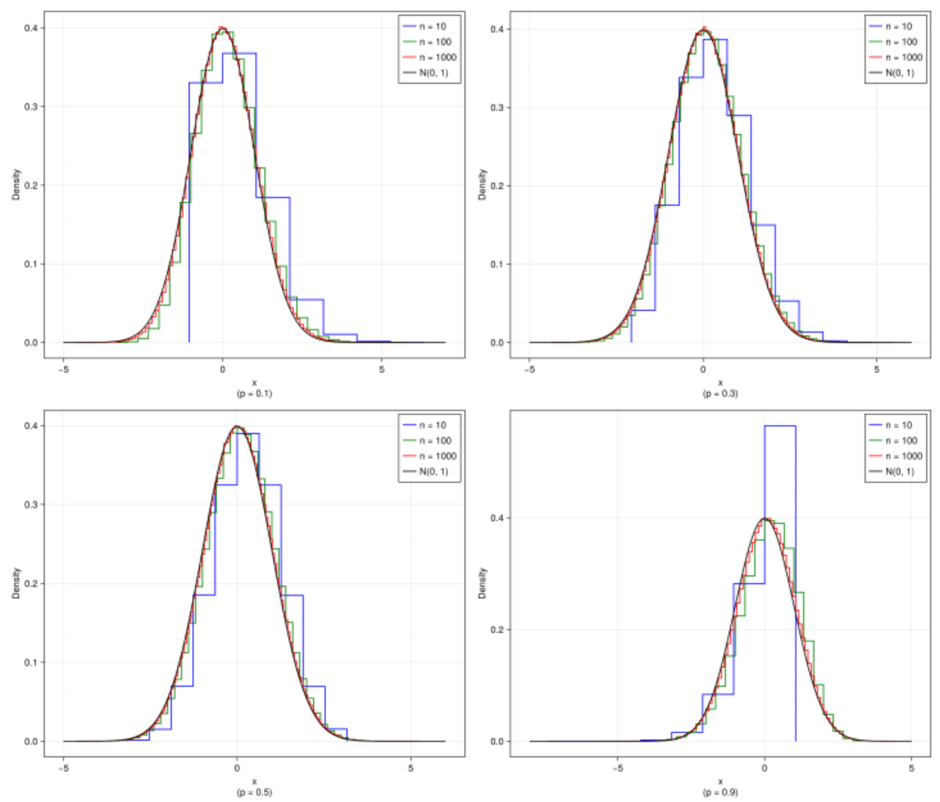
$$p(x) = P(X=x) = \begin{cases} p, & \text{for } x=1 \\ 1-p, & \text{for } x=0 \end{cases}$$

and $E(X) = \sum_{i=1}^{2} x_i p(x_i) = p$, $Var(X) = E[(X-E(X))^2] = p(1-p)$.

Suppose $X_1, X_2, \ldots, X_n$ are n i.i.d. Bernoulli random variables, let $Y_n = \sum_{i=1}^{n} X_i$

Based on the central limit theorem, $Y_n \underset{\text{appox}}{\sim} N(np, np(1-p))$, and $\tilde{Y}_n = \frac{Y_n - np}{\sqrt{np(1-p)}} \underset{\text{approx}}{\sim} N(0,1)$ as $n \to \infty$.

standardization



Evidently, independent of p, as $n \to \infty$, the Binomial distribution converges to the normal distribution, but for p approaching 0.5, smaller n also works fine in comparison with $p \to 1$ or 0.

Since the Binomial distribution is discrete and the Normal distribution is continuous, the lower cutoff should subtract 0.5 and the upper cutoff should add 0.5 when estimating the probability of the Binomial distribution using the Normal one.

e.g. $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$P(X=k)$ = the integral area between $k-0.5$ and $k+0.5$ of the corresponding Normal distribution

The above is called the continuity correction.

Poisson approximation

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The PDF of the Poisson distribution is

$$p(k) = P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,\dots$$

with parameter λ, a positive number. $E(X)=\lambda$, $Var(X)=\lambda$.

The PDF of the Binomial distribution is

$$p(k) = P(X=k) = \binom{n}{k}p^k(1-p)^{n-k}, \quad k=0,1,\dots,n$$

with parameters n and p, denoting the number of experiments and the probability of success, respectively. $E(X)=np$, $Var(X)=np(1-p)$.

Let $\lambda = np \Rightarrow p = \frac{\lambda}{n}$, we have for the Binomial distribution

$$p(k) = P(X=k) = \frac{n!}{(n-k)!\,k!}p^k(1-p)^{n-k}$$

$$= \frac{n\cdot(n-1)\cdot\dots\cdot(n-k+1)}{k!}\frac{\lambda^k}{n^k}\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n}{n}\cdot\frac{n-1}{n}\cdot\dots\cdot\frac{n-k+1}{n}\cdot\frac{\lambda^k}{k!}\cdot\left(1-\frac{\lambda}{n}\right)^n\cdot\left(1-\frac{\lambda}{n}\right)^{-k}$$

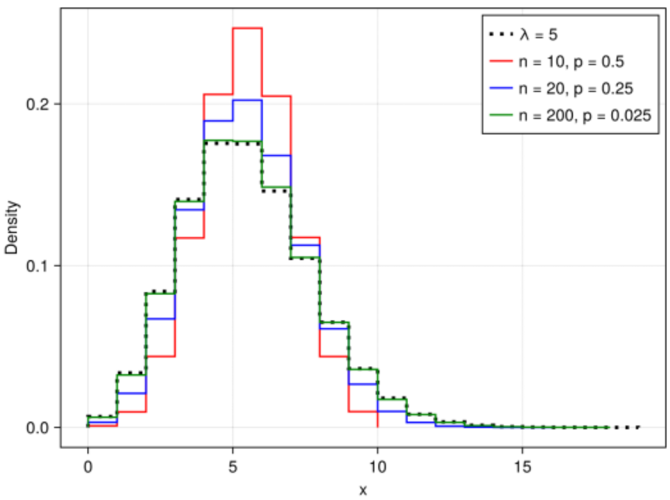
For a fixed k and let $n\to\infty$, we have

$$\frac{n-1}{n},\dots,\frac{n-k+1}{n}\to 1, \quad p=\frac{\lambda}{n}\to 0, \quad \left(1-\frac{\lambda}{n}\right)^{-k}\to 1, \quad \left(1-\frac{\lambda}{n}\right)^n\to e^{-\lambda}$$

Therefore, for each fixed k, as $n\to\infty$, $p\to 0$, and $\lambda=np$ a constant, we have

$$\binom{n}{k}p^k(1-p)^{n-k}\longrightarrow e^{-\lambda}\frac{\lambda^k}{k!}$$

i.e. the Poisson distribution with parameter λ is the limit of the Binomial distribution with parameters n and p, which satisfy $\lambda=np$ as $n\to\infty$ and $p\to 0$.



Evidently, as $n\to\infty$, $p\to 0$, the Binomial distribution with parameters n and p converges to the Poisson distribution with parameter $\lambda=np$.